ENTROPY-EXPANSIVENESS FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS.

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ABSTRACT. We show that diffeomorphisms with a dominated splitting of the form $E^s \oplus E^c \oplus E^u$, where E^c is a nonhyperbolic central bundle that splits in a dominated way into 1-dimensional subbundles, are entropy-expansive. In particular, they have a principal symbolic extension and equilibrium states.

1. INTRODUCTION

In dynamical systems one often considers the following three main levels of structure: measure theoretic, topological, and infinitesimal (properties of the derivative). Connections between such different levels have always been of high interest. For example, uniform hyperbolicity, an infinitesimal property, implies a rich structure on the other two levels. This paper is part of a program that studies how more general infinitesimal properties (partial hyperbolicity and existence of dominated splittings) force a certain topological and measure-theoretic behavior for the underlying dynamics. Here we will focus on a special type of partial hyperbolicity that will ensure the system is entropy-expansive.

A diffeomorphism f is α-expansive, $\alpha > 0$, if $\operatorname{dist}(f^n(x), f^n(y)) \leq \alpha$ for all $n \in \mathbb{Z}$ implies $x = y$. Uniform hyperbolicity implies α-expansiveness for some $\alpha > 0$. One can relax this condition requiring entropy-expansiveness. This notion, introduced by Bowen [B], is characterized by the fact that, for every small $\alpha > 0$ and every point $x \in M$, the intersection of the sets $f^{-n}(B(f^n(x), \alpha))$, $n \in \mathbb{Z}$, has zero topological entropy. Here $B(x, \alpha)$ is the ball centered at x of radius α.

Entropy-expansive maps are not necessarily expansive, but have similar properties to expansive maps in regards to topological and measure theoretic entropy. For instance, entropy-expansive maps always have equilibrium states, [K], and symbolic extensions preserving the entropy structure (called principal extensions), [BFF]. For a broad discussion between these notions see [DN, Section 1].

Several results illustrate the interplay between smoothness and entropy-expansive-like properties. First, it follows from [Bz] and [BFF] that C^∞ diffeomorphisms are asymptotically h-expansive. In [DM] it was shown that every C^2 interval map has a symbolic extension. A similar result for C^2 surface diffeomorphisms can be found in [Br]. These results support the conjecture of Downarowicz and Newhouse [DN] that every C^2 diffeomorphism has a symbolic extension. However, this conjecture does not hold for C^1 diffeomorphisms in any manifold of dimension three or higher, [As, DF].

In this paper we adopt a different approach and study the relation between "hyperbolic-like properties" and entropy-expansiveness. Indeed uniformly hyperbolic diffeomorphisms are entropy-expansive. There are also some results available for "weakly hyperbolic" systems. For instance, in [PV$_1$] for a surface diffeomorphisms f and a compact f-invariant set Λ with a dominated splitting it is shown that the map f restricted to Λ is entropy-expansive. See also further related results in [PV$_2$]. Finally, in [CY] it is shown that every partially hyperbolic set with a one-dimensional center direction is entropy-expansive.

Here we continue with the above investigations and consider partially hyperbolic sets whose center bundle is higher dimensional, but splits in a dominated way into one-dimensional subbundles. We prove that such diffeomorphisms are entropy-expansive:

Theorem 1.1. *Let f be a diffeomorphism and Λ be a compact f-invariant set admitting a dominated splitting $E^s \oplus E_1 \oplus \cdots \oplus E_k \oplus E^u$, where E^s is uniformly contracting, E^u is uniformly expanding, and all E_i are one-dimensional. Then $f|_\Lambda$ is entropy-expansive.*

In the previous theorem we allow for the bundles E^s and E^u to possibly be empty.

Observe that the conditions in the theorem are also necessary for entropy-expansiveness at least for C^1 generic diffeomorphisms. More precisely, in contrast with our result, C^1 generically diffeomorphisms having a central (non-hyperbolic) indecomposable bundle of dimension at least two are not entropy expansive, [DF, As]. In fact, the proof of these results follows the methods introduced in [DN] relating homoclinic tangencies to non-existence of symbolic extensions. Roughly, the

existence of an indecomposable central of dimension two (or higher) leads to the appearance of persistent homoclinic tangencies which in turns prevent entropy-expansiveness. We observe that the hypotheses of Theorem 1.1 prevents the creation of homoclinic tangencies by perturbations, see for instance [W].

Next we derive some consequences of Theorem 1.1. In [BFF] it is shown that every entropy-expansive diffeomorphism has a principal symbolic extension. We then have the next corollary.

Corollary 1.2. *If Λ and f are as in Theorem 1.1, then $f|_\Lambda$ has a principal symbolic extension*

Since every entropy-expansive diffeomorphism has an equilibrium state we have the next result.

Corollary 1.3. *For Λ and f as in Theorem 1.1, if $\varphi \in C^0(\Lambda)$, then $f|_\Lambda$ has an equilibrium state associated with φ.*

As domination is a key ingredient in our constructions we have the next natural question.

Question 1.4. *Let f be a diffeomorphism and Λ be a compact f-invariant set with a Df-invariant splitting (not necessarily dominated) $T_\Lambda M = E^s \oplus E_1 \oplus \cdots \oplus E_k \oplus E^u$, with E^s uniformly contracting, E^u uniformly expanding, and $E_1, \ldots, E_k$ one-dimensional. Is f entropy-expansive?*

We note that as we were preparing this paper Liao, Viana, and Yang [LVY] announced that diffeomorphisms far from homoclinic tangencies satisfy Shub's Entropy Conjecture, see [S], and have a principal symbolic extension. This conjecture relates the topological entropy to the spectral radius of the action induced by the system on the homology (see previous partial results in [SX]).

This paper is organized as follows. In Section 2 we provide background, including the existence of fake foliations. In Section 3 we prove Theorem 1.1.

2. Definitions and background

We now recall the main concepts in this paper; namely, the notions of entropy-expansiveness and dominated splittings.

2.1. **Entropy and symbolic extensions.** In what follows (X, d) is a compact metric space and f is a continuous self-map of X. The d_n metric on X is defined as

$$d_n(x, y) := \max_{0 \leq i \leq n-1} \text{dist}(f^i(x), f^i(y))$$

and is equivalent to d and defined for all $n \geq 0$.

For a set $Y \subset X$, a set $A \subset Y$ is (n, ϵ)-*spanning* if for any $y \in Y$ there exists a point $x \in A$ where $d_n(x, y) < \epsilon$. The minimum cardinality of the (n, ϵ)-spanning sets of Y is denoted $r_n(Y, \epsilon)$. We let

$$(1) \quad \bar{r}(Y, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log r_n(Y, \epsilon) \quad \text{and} \quad \tilde{h}(f, Y) := \lim_{\epsilon \to 0} \bar{r}(Y, \epsilon).$$

To see that the last limit exists see for instance [M]. The *topological entropy* $h_{\text{top}}(f)$ of f is $\tilde{h}(f, X)$.

Given $\epsilon > 0$ let

$$\Gamma_\epsilon^+(x) := \bigcap_{n=0}^\infty f^{-n}(B_\epsilon(f^n(x)))$$

and set

$$h_f^*(\epsilon) := \sup_{x \in X} \tilde{h}(f, \Gamma_\epsilon^+(x)).$$

The map f is *entropy-expansive,* or *h-expansive* for short, if there exists some $c > 0$ such that $h_f^*(\epsilon) = 0$ for all $\epsilon \in (0, c)$.

If f is a homeomorphism, then we define

$$\Gamma_\epsilon(x) : = \bigcap_{n \in \mathbb{Z}} f^{-n}(B_\epsilon(f^n(x))) \text{ and } h_{f, \text{homeo}}^*(\epsilon) : = \sup_{x \in X} \tilde{h}(f, \Gamma_\epsilon(x)).$$

If X is a compact space and f is a homeomorphism, then $h_f^*(\epsilon) = h_{f, \text{homeo}}^*(\epsilon)$, [B].

For an f-invariant measure μ the *measure theoretic entropy of* f measures the exponential growth of orbits under f that are "relevant" to μ and is denoted $h_\mu(f)$, see for instance [KH] for a precise definition. The *variational principle* states that if X is a compact metric space and f is continuous, then $h_{\text{top}}(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f)$, where $\mathcal{M}(f)$ is the space of all invariant Borel probability measures for f.

If f is a homeomorphism and $\varphi \in C^0(X)$, then the *pressure of* f *with respect to* φ *and* $\mu \in \mathcal{M}(f)$ is

$$P_\mu(\varphi, f) := h_\mu(f) + \int \varphi d\mu.$$

The topological pressure of (X, f), denoted $P(\varphi, f)$, corresponds to a "weighted" topological entropy, see [KH, p. 623]. The *variational*

principle for pressure states that if f is a homeomorphism of X and $\varphi \in C^0(X)$ then

$$P(\varphi, f) := \sup_{\mu \in \mathcal{M}(f)} P_\mu(\varphi, f).$$

A measure μ such that $P(\varphi, f) = P_\mu(\varphi, f)$ is called an *equilibrium state*.

A dynamical system (X, f) has a *symbolic extension* if there exists a subshift (Y, σ) and a continuous surjective map $\pi : Y \to X$ such that $\pi \circ \sigma = f \circ \pi$. The system (Y, σ) is called an *extension* of (X, f) and (X, f) is called a *factor* of (Y, σ). Note that the subshift need not be of finite type and the factor map may be infinite-to-one. A nice form of a symbolic extension is a *principal extension*, that is, an extension given by a factor map which preserves entropy for every invariant measure, see [BD].

2.2. **Dominated splittings.**

Through the rest of the paper we assume that M is a finite dimensional, smooth, compact, and boundaryless Riemannian manifold and $f : M \to M$ is a C^1 diffeomorphism. An f-invariant set Λ (not necessarily closed) has a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E \oplus F$ such that

    (i) the bundles E and F are both non-trivial,
    (ii) the fibers $E(x)$ and $F(x)$ have dimensions independent of $x \in \Lambda$, and
    (iii) there exist $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n|E(x)\| \cdot \|Df^{-n}|F(f^n(x))\| \leq C\lambda^n,$$

       for all $x \in \Lambda$ and $n \geq 0$.

More generally, a Df-invariant splitting

$$T_\Lambda M = E_1 \oplus \cdots \oplus E_k$$

is dominated if for all $i = 1, \ldots, (k-1)$ the splitting $T_\Lambda M = E_1^i \oplus E_{i+1}^k$ is dominated, where $E_j^\ell = E_j \oplus \cdots \oplus E_\ell$, for $1 \leq j \leq \ell \leq k$.

Note that the above definitions imply the continuity of the splittings.

We consider a diffeomorphism f and a compact f-invariant set Λ with a dominated splitting $E^s \oplus E_1 \oplus \cdots \oplus E_k \oplus E^u$ as in Theorem 1.1. For $x \in \Lambda$ and $i \in \{1, ..., k\}$ let us denote

$$(2) \qquad \begin{aligned} E^{cs,i}(x) &:= E^s(x) \oplus E_1(x) \oplus \cdots \oplus E_i(x) \text{ and} \\ E^{cu,i}(x) &:= E_i(x) \oplus \cdots \oplus E_k(x) \oplus E^u(x). \end{aligned}$$

We also let $E^{cs,0} = E^s$ and $E^{cu,k+1} = E^u$ and write $s = \dim(E^s)$ and $u = \dim(E^u)$.

By definition $E^{cs,i}(x) \oplus E^{cu,i+1}(x)$ is a dominated splitting for Λ and

$$\|Df^n|_{E^{cs,i}(x)}\| \cdot \|Df^{-n}|_{E^{cu,i+1}(x)}\| \leq C\lambda^n$$

for some $C \geq 1$ and $\lambda \in (0,1)$, and all $i \in \{0, ..., k\}$, $x \in \Lambda$, and $n \geq 0$.

The next proposition is an immediate consequence of [G, Theorem 1] and will simplify many of the arguments.

Proposition 2.1. *There exists an adapted Riemannian metric $\|\cdot\|_0$, equivalent to the original one and $\lambda \in (0,1)$ such that*

$$(3) \qquad \prod_{j=0}^{n} \|Df|E^{cs,i}(f^j(x))\|_0 \cdot \|Df^{-1}|E^{cu,i+1}(f^j(x))\|_0 < \lambda^n$$

for all $n \geq 0$, all $x \in \Lambda$, and all $i \in \{0, ..., k\}$.

Proof. Fix $i \in \{0, ..., k\}$. By [G] there exist an adapted Riemannian metric equivalent to the original one and $\lambda_i \in (0,1)$ such that $\|Df|E^{cs,i}(x)\|_0 \cdot \|Df^{-1}|E^{cu,i+1}(f(x))\|_0 \leq \lambda_i$. Since the splitting is invariant we conclude that

$$\prod_{j=0}^{n} \|Df|E^{cs,i}(f^j(x))\|_0 \cdot \|Df^{-1}|E^{cu,i+1}(f^j(x))\|_0 < \lambda_i^n.$$

Letting $\lambda = \max\{\lambda_0, ..., \lambda_k\}$ we prove the lemma. $\qquad \square$

Throughout the rest of the paper we assume that the Riemannian metric is an adapted metric.

For each $i = 0, \ldots, k$, consider the dominated splitting $E^{cs,i} \oplus E^{cu,i+1}$. If $V(\Lambda)$ is a small neighborhood of Λ then the f-invariant set

$$(4) \qquad \qquad \Lambda_V = \bigcap_{n \in \mathbb{Z}} f^n(\overline{V(\Lambda)})$$

has a dominated splitting that extends the splitting on Λ, see for instance [BDV, App. B]. We also denote these extensions by $E^{cs,i}$ and $E^{cu,i+1}$. Moreover, these splittings can be continuously extended to $V(\Lambda)$. These extensions are "nearly" invariant under f. That is, there are sufficiently small cone fields about the extended splitting that are invariant. We denote these extensions by $\widetilde{E}^{cs,i}$ and $\widetilde{E}^{cu,i+1}$ and the small cone fields by $\mathcal{C}(\widetilde{E}^{cs,i})$ and $\mathcal{C}(\widetilde{E}^{cu,i+1})$.

2.3. **Fake center manifolds.** Much of this section follows Section 3 of [BW]. The next proposition is similar to Proposition 3.1 in [BW].

Proposition 2.2. *Let $f : M \to M$ be a C^1 diffeomorphism and Λ a compact f-invariant set with a partially hyperbolic splitting,*

$$T_\Lambda M = E^s \oplus E_1 \oplus \cdots \oplus E_k \oplus E^u.$$

Let $E^{cs,i}$ and $E^{cu,i}$ be as in equation (2) and consider their extensions $\tilde{E}^{cs,i}$ and $\tilde{E}^{cu,i}$ to a small neighborhood of Λ.

Then for any $\epsilon > 0$ there exist constants $R > r > r_1 > 0$ such that, for every $p \in \Lambda$, the neighborhood $B(p,r)$ is foliated by foliations $\widehat{W}^u(p)$, $\widehat{W}^s(p)$, $\widehat{W}^{cs,i}(p)$, and $\widehat{W}^{cu,i}(p)$, $i \in \{1, ..., k\}$, such that for each $\beta \in \{u, s, (cs,i), (cu,i)\}$ the following properties hold:

  (i) Almost tangency of the invariant distributions. For each $q \in B(p,r)$, the leaf $\widehat{W}_p^\beta(q)$ is C^1, and the tangent space $T_q\widehat{W}_p^\beta(q)$ lies in a cone of radius ϵ about $\widetilde{E}^\beta(q)$.
  (ii) Coherence. $\widehat{W}_p^s$ subfoliates $\widehat{W}_p^{cs,i}$ and $\widehat{W}_p^u$ subfoliates $\widehat{W}_p^{cu,i}$ for each $i \in \{1, ..., k\}$.
  (iii) Local invariance. For each $q \in B(p, r_1)$ we have

$$f(\widehat{W}_p^\beta(q, r_1)) \subset \widehat{W}_{f(p)}^\beta(f(q)) \ \text{ and } \ f^{-1}(\widehat{W}_p^\beta(q, r_1)) \subset \widehat{W}_{f^{-1}(p)}^\beta(f^{-1}(q)),$$

      here $\widehat{W}_p^\beta(q, r_1)$ is the connected component of $\widehat{W}_p^\beta(q) \cap B(q, r_1)$ containing q.
  (iv) Uniquencess. $\widehat{W}_p^s(p) = W^s(p, r)$ and $\widehat{W}_p^u(p) = W^u(p, r)$.

By choosing the neighborhood $V(\Lambda)$ of Λ above sufficiently small we have that Λ_V also satisfies the hypotheses in Proposition 2.2 and hence the points in Λ_V have fake foliations as in the proposition.

Remark 2.3. For ϵ sufficiently small, the transversality of the invariant bundles for Λ_V implies that, for all $p \in \Lambda_V$ and every x and y sufficiently close to Λ_V, then $\widehat{W}_p^{cs,i}(x) \cap \widehat{W}_p^{cu,i+1}(y)$ consists of a single point for all $i \in \{0, ..., k\}$. Here $\widehat{W}_p^s(x) = \widehat{W}^{cs,0}(x)$ and $\widehat{W}_p^u(y) = \widehat{W}_p^{cu,k+1}(y)$.

The proof of Proposition 2.2 is very similar to the one of Proposition 3.1 in [BW], inspired by Theorem 5.5 in [HPS]. So it is omitted. In [HPS] the result is that the leaves would be tangent at p to the invariant bundles, but the leaves do not form necessirely a foliation. In the proposition above, the fake foliations are not tangent to the initial bundles but they stay within thin cone fields. The main advantage is that these fake leaves foliate local neighborhoods. As in [HPS] to get the fake foliations one use a graph transform.

2.4. **Central curves.** Throughout the rest of the paper we fix $\rho > 0$ such that for all $i \in \{0, ..., k\}$ and $x \in V(\Lambda)$ there exists a curve $\gamma_i(x)$ centered at x of radius ρ tangent to the bundle E_i.

The next lemma is a higher dimensional version of [PV$_1$, Lemma 2.2]. Since the proof is analogous to the one there we omit it.

Lemma 2.4. *Let Λ be and f-invariant set as in Theorem 1.1. For any sufficiently small $\rho > 0$ and any $\delta \in (0, \rho)$, if $x \in \Lambda_V$, $y \in \Gamma_\delta(x)$, and γ is a curve with endpoints x and y that is contained in $B_\rho(x)$ and tangent to E_i for some $i \in \{1, ..., k\}$ then*

$$\ell(\gamma) < 2\,\delta \ \text{ and } \ \gamma \subset \Gamma_{2\,\delta}(x).$$

3. Proof of Theorem 1.1

We now proceed to prove our main theorem. The idea of the proof is that using the fake foliations we can show that the set $\Gamma_\epsilon(x)$ is 1-dimensional for each point. Under iteration this set will stay in a 1-dimensional set of finite radius. Then a folklore fact will show that the set $\Gamma_\epsilon(x)$ has zero topological entropy. As we can do this uniformly we will know that Λ is entropy expansive with for f.

3.1. **Hyperbolic-like behavior of fake foliations.**

Remark 3.1 (Choice of constants)**.** We fix some constants:

(a) Fix $\tau > 0$ such that $(1 + \tau)\sqrt{\lambda} < 1$, where $\lambda < 1$ is the domination constant in (3).

(b) Fix $\nu > 0$ sufficiently small such that $\bigcup_{x \in \Lambda} B_{5\nu}(x) \subset V(\Lambda)$ and such that if $y, y' \in B_{5\nu}(x)$ for some $x \in \Lambda$, then for all $i \in \{0, ..., k\}$ it holds

$$1 - \tau < \frac{\|Df^{-1}|_{\widetilde{E}^{cu,i}(y)}\|}{\|Df^{-1}|_{\widetilde{E}^{cu,i}(y')}\|} < 1 + \tau \quad \text{and} \quad 1 - \tau < \frac{\|Df|_{\widetilde{E}^{cs,i}(y)}\|}{\|Df|_{\widetilde{E}^{cs,i}(y')}\|} < 1 + \tau.$$

To obtain the "1-dimensionality" of the sets $\Gamma_\delta(x)$ we use hyperbolic times. To do so the first step is the following reformulation of Pliss Lemma stated for the sets Λ_V satisfying the hypotheses of Theorem 1.1.

Lemma 3.2. *[Al, P] Let $\lambda > 0$ be as in Proposition 2.1 and $0 < \lambda < \lambda_1 < \lambda_2 < 1$. Assume that $x \in \Lambda_V$, $i \in \{0, ..., k\}$, and there exists $n \geq 0$ such that*

$$(5) \qquad \prod_{m=0}^{n} \|Df|_{E^{cs,i}(f^m(x))}\| \leq \lambda_1^n.$$

Then there is $N = N(\lambda_1, \lambda_2, f) \in \mathbb{N}$ and a constant $c = c(\lambda_1, \lambda_2, f) > 0$ such that for every $n \geq N$ there exist $\ell \geq c\,n$ and numbers (hyperbolic times)

$$0 < n_1 < n_2 < \cdots < n_\ell < n$$

such that

$$\prod_{m=n_r}^{h} \|Df|_{E^{cs,i}(f^m(x))}\| \leq \lambda_2^{h-n_r},$$

for all $r = 1, 2, \ldots, \ell$ and all h with $n_r \leq h \leq n$.
  Similar assertions hold for the map f^{-1} and the bundles $E^{cu,i}$.

The next application of Lemma 3.2 provides a lower bound for the expansion of Df along the bundle $E^{cs,i}$ and of Df^{-1} along the bundle $E^{cu,i}$. Given a central curve $\gamma_i(x)$ and points $y, z \in \gamma_i(x)$ we let $[y, z]_{\gamma_i(x)}$ be the segment in $\gamma_i(x)$ with endpoints y, z.

Lemma 3.3. *Consider a small enough $\delta > 0$. If $x \in \Lambda_V$, $y \in \Gamma_\delta(x)$, $y \neq x$, $i \in \{1, ..., k\}$. Suppose that $y \in \gamma_i(x)$. Then $[x, y]_{\gamma_i(x)} \subset \Lambda_V$ and if $\lambda_1 \in (\lambda, \sqrt{\lambda})$ then there is $n_0 > 0$ such that*

$$\prod_{j=0}^{n} \|Df|_{E^{cs,i}(f^j(f^{-n}(y')))}\| > \lambda_1^n \quad and \quad \prod_{j=0}^{n} \|Df^{-1}|_{E^{cu,i}(f^j(y'))}\| > \lambda_1^n$$

for all $y' \in [x, y]_{\gamma_i(x)}$ and $n > n_0$.

This lemma means that, for $\delta > 0$ sufficiently small, if $[x, y]_{\gamma_i(x)} \subset \Gamma_\delta(x)$ and $y \neq x$ then for all point $y' \in [x, y]_{\gamma_i(x)}$ the leaves of the fake foliations $\widehat{W}_x^{cs,i-1}(y')$ and $\widehat{W}_x^{cu,i+1}(y')$ behave like leaves of the stable and unstable foliations, respectively. Thus we have the following consequence:

Corollary 3.4. *Let $y \in \Gamma_\delta(x) \setminus \{x\}$ such that the central curve $[x, y]_{\gamma_i(x)}$ is contained in $\Gamma_\delta(x)$ for some small $\delta > 0$. Then for all $y' \in [x, y]_{\gamma_i(x)}$ we have that*

$$\widehat{W}_x^{cu,i+1}(y') \cap \Gamma_\delta(x) = \{y'\}. \quad and \quad \widehat{W}_x^{cs,i-1}(y') \cap \Gamma_\delta(x) = \{y'\}.$$

Proof of Lemma 3.3. By Lemma 2.4, if δ is sufficiently small then the curve $[x, y]_{\gamma_i(x)}$ is contained in Λ_V and thus the bundles $E^{cs,i}(y')$ and $E^{cu,i}(y')$ are defined along the orbit of any $y' \in [x, y]_{\gamma_i(x)}$.
  Let $\lambda_2 \in (\lambda_1, 1)$ such that $(1 + \tau)\lambda_2 < 1$, where τ is as in Remark 3.1 (b). Let us prove the first inequality. Arguing by contradiction, suppose that there exist infinitely many $m_n \in \mathbb{N}$, $m_n \to \infty$, and $y_n \in [x, y]_{\gamma_i(x)}$ such that

$$\prod_{j=0}^{m_n} \|Df|_{E^{cs,i}(f^j(f^{-m_n}(y_n)))}\| \leq \lambda_1^n.$$

Writing $w_n = f^{-m_n}(y_n)$, by Lemma 3.2 we have

$$\prod_{j=0}^{m_n} \|Df|_{E^{cs,i}(f^j(w_n))}\| \leq \lambda_2^{m_n}.$$

By Lemma 2.4, for all $j \geq 0$, the curve $f^{-j}([x, y]_{\gamma_i(x)})$ stays $2\,\epsilon$-close to $f^{-j}(x)$, and thus $2\,\epsilon$-close to $f^{-j}(w_n)$. It follows from Remark 3.1 (b) that, for all $y' \in f^{-m_n}([x, y]_{\gamma_i(x)})$, one has

$$\prod_{j=0}^{m_n} \|Df|_{E^{cs,i}(f^j(y'))}\| \leq \left((1+\tau)\,\lambda_2\right)^{m_n}.$$

Since $[x, y]_{\gamma_i(x)} = f^{m_n}\left([f^{-m_n}(x), f^{-m_n}(y)]_{\gamma_i(f^{-m_n}(x))}\right)$ we have that

$$\ell\left([x, y]_{\gamma_i(x)}\right) \leq \left((1+\tau)\,\lambda_2\right)^{m_n} \ell\left([f^{-m_n}(x), f^{-m_n}(y)]_{\gamma_i(f^{-m_n}(x))}\right).$$

By Lemma 2.4, if ϵ is small enough, $\ell\left([f^{-m_n}(x), f^{-m_n}(y)]_{\gamma_i(f^{-m_n}(x))}\right)$ is bounded by $2\,\delta$. Thus letting $m_n \to +\infty$ we get that $x = y$, a contradiction.

To get the other product we simply look to the bundle $E^{cu,i}$ and the map f^{-1} and repeat the above argument. □

3.2. **End of the proof of Theorem 1.1.** The main step of the proof of Theorem 1.1 is the following result.

Proposition 3.5. *For $x \in \Lambda$ and $\delta > 0$ sufficiently small the set $\Gamma_\delta(x)$ is either $\{x\}$ or is contained in a curve $\gamma_i(x)$ for some $i \in \{1, ..., k\}$.*

We postpone the proof of this proposition and prove the theorem.

Proof of Theorem 1.1. Let f and Λ satisfy the hypothesis of Theorem 1.1. Then from Proposition 3.5 we know that for $\delta > 0$ sufficiently small the set $\Gamma_\delta(x)$ is a single point or contained in a central curve $\gamma_i(x)$. It is a folklore fact that the entropy of 1-dimensional curves of bounded length is zero (for a proof see for instance [BFSV]). This implies that $\tilde{h}(f, \Gamma_\epsilon(x)) = 0$ for all x and every small ϵ. Hence, the set Λ is entropy expansive for f. □

Proof of Proposition 3.5. Fix $x \in \Lambda$ and assume that there is $y \in \Gamma_\delta(x) \setminus \{x\}$. We start with the following lemma.

Lemma 3.6. *For every small enough $\delta > 0$,*

$$\Gamma_\delta(x) \subset \widehat{W}_x^{cs,k}(x) \cap \widehat{W}_x^{cu,0}(x).$$

Proof. We see that $\Gamma_\delta(x) \subset \widehat{W}_x^{cs,k}(x)$, the inclusion $\Gamma_\delta(x) \subset \widehat{W}_x^{cu,0}(x)$ follows similarly. Take $x \in \Gamma_\delta(x)$. If $y \notin \widehat{W}_x^{cs,k}(x)$, then as $\widetilde{E}^u$ is uniformly expanding, after forward iterations the orbit of y will scape from $\widehat{W}_x^{cs,k}(x)$ and thus from x, contradicting that $y \in \Gamma_\delta(x)$. This ends the proof of the lemma. □

Given $j \in \{1, \ldots, k\}$, using Proposition 2.2, we consider small r and the submanifold

$$(6) \qquad \widetilde{W}^{cs,j}(x) = \bigcup_{z \in \gamma_j(x)} \widehat{W}^{cs,j-1}_x(z, r).$$

This submanifold has dimension $s + j$ and is transverse to $\widehat{W}^{cu,j+1}_x(z)$ for all z close to x. Note that $\widetilde{W}^{cs,1}(x)$ is foliated by stable manifolds (recall that $\widehat{W}^{cs,0}_x(z) \subset W^s(z)$).

For every $j \in \{1, \ldots, k\}$ and every $y \in \Gamma_\delta(x) \cap \widehat{W}^{cs,j}_x(x)$ we associate a pair of points $\widetilde{y}_j \in \widetilde{W}^{cs,j}(x)$ and $\overline{y}_j \in \gamma_j(x)$ defined as follows, see Figure 1,

$$(7) \quad \widetilde{y}_j \stackrel{\text{def}}{=} W^{cu,j+1}_x(y) \cap \widetilde{W}^{cs,j}(x), \quad \text{where} \quad \widetilde{y}_j = \widehat{W}^{cs,j-1}_x(\overline{y}_j) \cap \gamma_j(x).$$
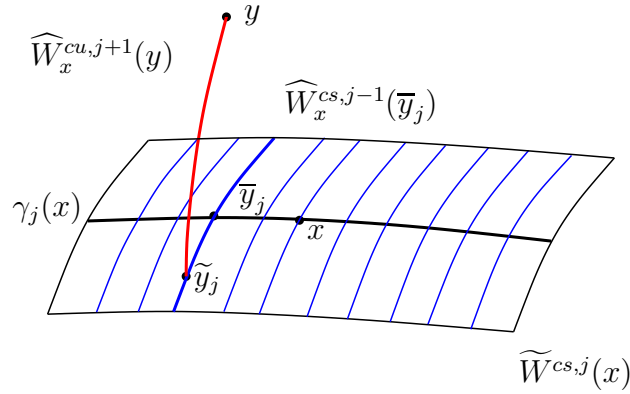

FIGURE 1. The points $\widetilde{y}_j$ and $\overline{y}_j$.

Claim 3.7. *Given small $\delta > 0$ and $y \in \Gamma_\delta(x) \cap \widehat{W}^{cs,j}_x(x)$ then*
    a) either $\overline{y}_j \neq x$,
    b) $\overline{y}_j = x$ and $\widetilde{y}_j \neq x$.

Proof. It is enough to see that the case $\overline{y}_j = \widetilde{y}_j = x$ can not occur. If so, by Remark 2.3, $y \in \widehat{W}^{cu,j+1}_x(x) \cap \widehat{W}^{cs,j}_x(x) = \{x\}$, which is a contradiction. $\qquad \square$

The next two lemmas follow straighforwardly from the fact that the angles between unitary vectors in the cone fields $\mathcal{C}(E^{cs,j})$ and $\mathcal{C}(E^{cu,j+1})$ are uniformly bounded away from zero.

Lemma 3.8. *There is $\kappa > 0$ such that for every $j \in \{1, \ldots, k\}$ and every $\delta > 0$ small enough the following property holds:*

For every $x \in \Lambda$, every $y \in B_\delta(x)$, every local submanifolds $N(x)$ of dimension $s+j$ tangent to the conefield $\mathcal{C}(E^{cs,j})$ containing x and $M(y)$ of dimension $(k-j)+u$ tangent to the conefield $\mathcal{C}(E^{cu,j+1})$ containing y one has that $N(x) \cap M(y)$ is contained $B_{\kappa\delta}(x)$.

Lemma 3.9. *There is $\kappa > 0$ such that for every $j \in \{1, \ldots, k\}$ and every $\delta > 0$ small enough the following property holds:*

Take any $x \in \Lambda$ and the local manifold $\widetilde{W}^{cs,j}(x)$ in (6). For every $y \in B_\delta(x) \cap \widetilde{W}^{cs,j}(x)$ one has that $\gamma_j(x) \cap \widehat{W}_x^{cs,j-1}(y)$ is contained in $B_{\kappa\delta}(x)$.

The next lemma is similar to the above, but is concerned with what happens inside a submanifold tangent to a conefield.

Lemma 3.10. *There is $\widehat{\kappa} > 0$ such that if $y \in \Gamma_\delta(x) \cap \widehat{W}_x^{cs,j}(x)$ then if $\delta > 0$ is small enough then $\overline{y}_j, \widetilde{y}_j \in \Gamma_{\widehat{\kappa}\delta}(x)$.*

Proof. For simplicity let us omit the subscrit j and just write $\widetilde{y}$ and $\overline{y}$. Take r_1 as in Proposition 2.2.

Claim 3.11. *If $\delta > 0$ is small enough then for all $i \geq 0$ it holds*

$$f^i(\widetilde{y}) \in \widehat{W}_{f^i(x)}^{cu,j+1}(f^i(y), r_1) \quad and \quad f^i(\widetilde{y}) \in \widehat{W}_{f^i(x)}^{cs,j-1}(f^i(\overline{y}), r_1)$$

and $f^i(\overline{y})$ is in a central curve $\gamma_j(x, r_1)$ centered at x of radious r_1 tangent to E_j and containing x.

Proof. The proof goes by induction on i. For $i = 0$, by Lemma 3.8, we have that $\mathrm{dist}(y, \widetilde{y}) < \kappa\delta < r_1$ and thus $\mathrm{dist}(\widetilde{y}, x) < (\kappa + 1)\varepsilon$. Hence, for small δ, Lemma 3.9 implies that $\mathrm{dist}(\widetilde{y}, \overline{y}) < \kappa(\kappa + 1)\delta < r_1$. By construction the point $\overline{y}$ is in a central curve $\gamma_j(x, r_1)$. This ends the first inductive step.

Assume that the induction hypothesis holds for all $i = 0, \ldots, m$. By Proposition 2.2, this implies

$$f^{m+1}(\widetilde{y}) \in f\big(\widehat{W}_{f^m(x)}^{cs,j-1}(f^m(\overline{y}), r_1)\big) \subset \widehat{W}_{f^{m+1}(x)}^{cs,j-1}(f^{m+1}(\overline{y})), \quad and$$

$$f^{m+1}(\widetilde{y}) \in f\big(\widehat{W}_{f^m(x)}^{cu,j+1}(f^m(y), r_1)\big) \subset \widehat{W}_{f^{m+1}(x)}^{cu,j+1}(f^{m+1}(y)).$$

As $f^m(\overline{y}) \in \gamma_j(f^m(x), r_1)$ we get that $f^{m+1}(\overline{y}) \in \gamma_j(f^{m+1}(x))$ and thus, recalling the definition of $\widetilde{W}^{cs,j}(f^{m+1}(x))$ in (6),

$$f^{m+1}(\overline{y}) = \widehat{W}_{f^{m+1}(x)}^{cu,j+1}(f^{m+1}(y)) \cap \widetilde{W}^{cs,j}(f^{m+1}(x)).$$

As $\mathrm{dist}(f^{m+1}(y), f^m(x)) < \delta$, we can apply the arguments in the step $i = 0$ to obtain the claim. $\qquad\square$

By Lemma 3.8, we have $\text{dist}(f^{m+1}(y), f^{m+1}(\widetilde{y})) < \kappa\,\delta$ and hence $\text{dist}(f^{m+1}(x), f^{m+1}(\widetilde{y})) < (\kappa + 1)\,\delta$. Lemma 3.9 now implies that $\text{dist}(f^{m+1}(\overline{y}), f^{m+1}(\widetilde{y})) < \kappa\,(\kappa + 1)\,\delta$. Hence

$$\text{dist}(f^{m+1}(x), f^{m+1}(\overline{y})) < 2\,(\kappa + 1)^2\,\delta.$$

Taking $\widehat{\kappa} = 2\,(\kappa + 1)^2$ we end the proof of Lemma 3.10. □

Lemma 3.12. *For every $\delta > 0$ small such that there is $y \in \Gamma_\delta(x) \setminus \{x\}$ there is $j_0 \in \{1, \ldots, k\}$ such that*

$$\gamma_{j_0}(x) \cap \left(\Gamma_{\kappa'\,\delta}(x) \setminus \{x\}\right) \neq \emptyset, \quad \text{where } \kappa' = \widehat{\kappa}^k.$$

Proof. The next claim is needed in the proof of the lemma.

Claim 3.13. *Let $j \in \{1, \ldots, k\}$ and $y \in \widehat{W}_x^{cs,j}(x) \cap \Gamma_\delta(x)$, $y \neq x$. Then there are two possibilities:*

   a) *either $\gamma_j(x)$ contains at least two points of $\Gamma_{\widehat{\kappa}\,\delta}(x)$,*
   b) *or there is $\widehat{y} \in \widehat{W}_x^{cs,j-1}(x) \cap \Gamma_{\widehat{\kappa}\,\delta}(x)$.*

Proof. If $y \in \gamma_j(x)$ we are done. Otherwise consider the points $\overline{y}_j$ and $\widetilde{y}_j$ defined in equation (7). If $\overline{y}_j \neq x$, by Lemma 3.10, $\overline{y}_j \in \gamma_j(x) \cap \Gamma_{\widehat{\kappa}\,\delta}(x)$ and we are also done. Otherwise, $\overline{y}_j = x$ and we take the point $\widetilde{y}_j \in \left(\Gamma_{\widehat{\kappa}\,\delta}(x) \setminus \{x\}\right)$, recall Claim 3.7. This point belongs to $\widehat{W}_x^{cs,j-1}(\overline{y}_j) = \widehat{W}_x^{cs,j-1}(x)$. Taking $\widehat{y} = \widetilde{y}_j$ one proves the claim. □

We are now ready to end the proof of Lemma 3.12. By Lemma 3.6 we have $y \in \widehat{W}_x^{cs,k}(x)$. Let $y \overset{\text{def}}{=} y_k$. Recursively, using the notation in Claim 3.13, for $j = 1, \ldots, k$, we define the points $y_{j-1} \overset{\text{def}}{=} \widetilde{y}_j$ where

$$(8) \qquad y_{j-1} = \widetilde{y}_j \in \widehat{W}_x^{cs,j-1}(x) \cap \Gamma_{\widehat{\kappa}^{k-j}\,\delta}(x), \quad y_{j-1} \neq x.$$

Since we are assuming that $\gamma_j(x) = \{x\}$ for all $j = 1, \ldots, k$, by Claim 3.13, the points y_j are well defined. By construction, we have $y_0 \in W^s(x)$, which is a contradiction. This proves the lemma. □

Scholium 3.14. The arguments in the proof of Lemma 3.12 implies the following. Assume that $\gamma_j(x) = \{x\}$ for $j = \iota + 1, \ldots, k$. Then to any point $y \in \Gamma_\delta(x)$ we can associate points $y_k = y, y_{k-1}, \ldots y_{\iota+1} \in \left(\Gamma_{\kappa'\,\delta}(x) \setminus \{x\}\right)$ such that

$$y_j \in \widehat{W}_x^{cs,j}(x) \cap \Gamma_{\kappa'\,\delta}(x).$$

End of the proof of Proposition 3.5. In view of Lemma 3.12, there is a largest $j \in \{1, \ldots, k\}$, that we denote by ι, such that $\gamma_\iota(x) \cap \Gamma_{\kappa'\,\delta}(x)$ contains at least two points. Then $\gamma_\iota(x) \cap \Gamma_{\kappa'\,\delta}(x) = [a, b]$, with $a \neq b$.

Since $\gamma_j(x) = \{x\}$, for $j = \iota + 1, \ldots, k$ we can consider the points $y_k = y$, $y_{k-1}, \ldots y_{\iota+1} \in \left(\Gamma_{\kappa'\delta}(x) \setminus \{x\}\right)$ satisfying Scholium 3.14.

For $z = y_{\iota+1}$ we let $\widetilde{z}_{\iota+1}, \overline{z}_{\iota+1} \in \Gamma_{\widehat{\kappa}\kappa'\delta}(x)$ as in Equation (7). Define

$$(9) \qquad\qquad \widetilde{y}^* = \widetilde{z}_{\iota+1}, \quad \overline{y}^* = \overline{z}_{\iota+1}.$$

By Claim 3.7 and by construction, there are two possibilities:

(1) $\widetilde{y}^* \in W_x^{cs,\iota-1}(\overline{y}^*)$, where $\overline{y}^* \in [a,b]$ and $\widetilde{y}^* \neq \overline{y}^*$, or
(2) $y_{\iota+1} \in W_x^{cu,\iota+1}(\overline{y}^*)$ and $\overline{y}^* \in [a,b]$.

Note that $\overline{y}^* \neq x$. As $[a,b]$ is non-trivial and $\overline{y}^* \in [a,b] \subset \Gamma_{\widehat{\kappa}\kappa'\delta}(x)$, Corollary 3.4 implies that, if case (1) holds then $\widetilde{y}^* \notin \Gamma_{\widehat{\kappa}\kappa'\delta}(x)$. Similarly, if case (2) holds, $y_{\iota+1} \notin \Gamma_{\widehat{\kappa}\kappa'\delta}(x)$. In both cases we get a contradiction, ending the proof of the proposition.

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